APPROXIMATE SOLUTIONS AND ASYMPTOTIC EXPANSIONS FOR THE PROBLEM OF BOUNDARY LAYER DEVELOPMENT DURING ACCELERATION

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The problem of boundary layer development around a body as it begins to move through an incompressible viscous fluid at rest is one of the basic problems of boundary layer theory. We prove the existence and uniqueness of the solution of this problem under certain natural conditions, use the method of straight lines to obtain approximate solutions and prove their convergence, and obtain expansions in powers of t for the quantity defining the resistance of the medium to the moving body and for certain other quantities. These expansions contain an arbitrary number of terms and are asymptotic as $t \to 0$. The remainder terms are estimated. Boundary layer development during acceleration was investigated by Blasius [1], Görtler [2], et al. Their studies are summarized in monograph [3].

Let us consider the problem of boundary layer development during acceleration for an external flow of the form $U(t, x) = t^n U_1(t, x)$ for any number $n \geqslant 1$, where $U_1(t, x)$ is either independent of t (as with Blasius and Görtler, who considered integer n) or is such that U_{1t} / U_1 is a bounded function.

The problem of boundary layer development at a body as it begins to move in a viscous incompressible fluid at rest in the case of symmetric flow leads us to consider the system of equations

$$u_t + uu_x + vu_y = -p_x + \nu u_{yy}, \qquad u_x + v_y = 0 \tag{1}$$

in the domain $D\{0 \leqslant t \leqslant T,\ 0 \leqslant x \leqslant X,\ 0 \leqslant y < \infty\}$ under the conditions

$$u|_{t=0} = 0, \quad u|_{x=0} = 0, \quad u|_{y=0} = 0, \quad v|_{y=0} = v_0(t, x), \quad u \to U \quad \text{for } y \to \infty \tag{2}$$

Here

$$U_t + UU_x = -p_x, \quad U|_{x=0} = 0, \quad U|_{t=0} = 0, \quad U > 0 \quad \text{for} \quad tx > 0$$

$$U = t^n U_1(t, x), \quad n \geqslant 1$$

The ratio U_{1t} / U_1 is a bounded function.

In [4] we showed that for $n = 1$ problem (1), (2) has a solution given a certain smoothness of the functions $U(t, x), v_0(t, x)$ and that this solution is unique in the domain D for $t \leqslant t_1$, where $t_1 = \text{const} > 0$, and depends on U and v_0. We also wrote out the first terms of the expansion of the function u in powers of t and estimated the remainder term. In addition, we proved that the approximate solutions obtained by solving a certain system of ordinary differential equations converge to the exact solution of problem (1), (2).

We shall carry out a similar analysis for the case of an external flow of the form $U = t^n U_1(t, x)$ for any $n \geqslant 1$. In addition, we shall construct an expansion in powers of t asymptotic as $t \to 0$ with an arbitrary number of terms and estimate the remainder term for the quantity $u_y(t, x, 0)$ which represents the resistance of the medium to the motion of the body, and for certain other quantities.

Specifically, in Görtler's case ($U(t, x) = t^n U_1(x)$, $n \geqslant 1$, $v_0(t,x) \equiv 0$) we have

$$u_y|_{y=0} = U_1(x)\, t^{n-1/2} \sum_{i=0}^{q} Y_i(x, 0)\, t^{i(n+1)} + U_1(x)\, O(t^{n-1/2+(q+1)(n+1)})$$

where q is an integer and where $Y_i(\xi, \eta)$, $i = 1, \ldots, q$ must be determined successively as the solutions of ordinary differential equations with respect to η which depend on the parameter ξ ; the quantity Y_0 does not depend on ξ (see Eqs. (20) with conditions (21), (22)).

An entirely similar procedure can be used to investigate boundary layer development during acceleration for three-dimensional axisymmetric flow, with the second equation of system (1) replaced by the equation

$$(r(t, x)u)_x + (r(t, x)\, v)_y = 0$$

where $r(t, x)$ is a given function describing the streamlined surface. The same applies to the problem of extension of the boundary layer when instead of the condition $u|_{x=0} = 0$ we are given the initial velocity profile

$$u|_{x=0} = u_1(t, y), \qquad u_1(t, y) > 0 \qquad \text{for } t, y > 0$$

The solution of problem (1), (2) can be constructed by the method applied in [4] to the study of problem (1), (2) in the case $n = 1$.

Let us make the following substitution of the independent coordinates in system (1):

$$\tau = t^{n-1/2}, \qquad \xi = x, \qquad \eta = \frac{u}{U} \tag{3}$$

Let us also introduce the new function

$$w = \frac{u_y t^n}{U} \tag{4}$$

Eliminating v from system (1), we obtain an equation of the form

$$\nu w^2 w_{\eta\eta} - \tau^3 (n - 1/2)\, w_\tau - \eta U \tau^{2N} w_\xi + n(\eta - 1)\, \tau^2 w_\eta +$$
$$+ A_1 \tau^{2N} w_\eta + B_1 \tau^{2N} w = 0 \qquad \left(N = 1 + \frac{1}{2n-1}\right) \tag{5}$$

for w in the domain $\Omega\{0 \leqslant \tau \leqslant T^{n-1/2}, 0 \leqslant \xi \leqslant X, 0 \leqslant \eta < 1\}$. In this expression $A_1 = (\eta^2 - 1)\, U_x + (\eta - 1)\, U_{1t} / U_1$, $\quad B_1 = -\eta U_x - U_{1t} / U_1$

We add to this the boundary conditions

$$w|_{\eta=1} = 0, \quad w|_{\tau=0} = 0, \quad (\nu w w_\eta - v_0 w \tau^N + n\tau^2 + C_1 \tau^{2N})|_{\eta=0} = 0 \tag{6}$$

where

$$C_1 = U_x + U_{1t} / U_1$$

Let us construct the solution of problem (5), (6) for some interval $0 \leqslant \tau \leqslant \tau_1$, $\tau_1 = \text{const} > 0$, and then obtain the solution of problem (1), (2) as a corollary.

The existence and uniqueness of the solution of problem (1), (2) for $n \geqslant 1$ can be proved essentially as in the case $n = 1$. We shall therefore merely point out the differences between the two proofs.

Let $f^{m,k}(\eta) \equiv f(mh, kh, \eta)$ for any function $f(\tau, \xi, \eta)$, $h = \text{const} > 0$. Let us consider the following system of ordinary differential equations in the interval [0, 1] of values of η :

$$L_{m,k}(w) \equiv \nu (w^{m,k})^2 w_{\eta\eta}^{m,k} - h^{-1}(mh)^3 (n - 1/2)(w^{m,k} - w^{m-1,k}) -$$

$$- h^{-1} \eta U_1^{m-1,k} ((m-1)\, h)^{3N} (w^{m,k} - w^{m,k-1}) + n(\eta - 1)(mh)^2 w_\eta^{m,k} + \tag{7}$$

(cont.)

$$+ A_1^{m-1,k}((m-1)h)^{2N} w_\eta^{m,k} + B_1^{m-1,k}((m-1)h)^{2N} w^{m,k} = 0 \quad \left(N = 1 + \frac{1}{2n-1}\right)$$

$$w^{0,k} = 0, \qquad m = 1, \ldots; \ k = 0, 1, \ldots, [X/h]$$

under the boundary conditions

$$w^{m,k}(1) = 0, \quad \lambda_{m,k}(w) \equiv [\nu w^{m,k} w_\eta^{m,k} - v_0^{m-1,k}((m-1)h)^N w^{m,k} +$$

$$+ n(mh)^2 + C_1^{m-1,k}((m-1)h)^{2N}]\,|_{\eta=0} = 0 \tag{8}$$

Let us show that as $h \to 0$ the solutions $w^{m,k}(\eta)$ of system (7) under conditions (8) converge for $mh \leqslant \tau_1 = \text{const} > 0$ to the solution of problem (5), (6). We begin by proving that the solution $w^{m,k}$ of problem (7), (8) exists for $mh \leqslant \tau_0$; we shall then establish estimates for this solution which are uniform in h.

The approximate solution

$$u^{m,k} = u\left((mh)^{\frac{N}{n}}, \ kh, \ \eta\right), \quad N = \left(1 + \frac{1}{2n-1}\right)$$

of problem (1), (2) can be obtained with the aid of the solution of problem (7), (8) from the formula

$$y = (mh)^N \int_0^{W^{m,k}} (w^{m,k}(s))^{-1} ds, \quad W^{m,k} = u^{mk} / U((mh)^N, \ kh)$$

The following lemma on the existence of the solution of problem (7), (8) can be justified exactly as in [4].

Lemma 1. Let U_1, U_{1x}, U_{1t} U_1^{-1}, v_0 be bounded in D. System (7) with boundary conditions (8) then has a solution $w^{m,k}(\eta)$ positive for $0 \leqslant \eta < 1$ if $0 \leqslant kh \leqslant X$ and $0 \leqslant mh \leqslant \tau_0 \leqslant T^{n-1/2}$, where τ_0 depends on U and v_0. The functions $w_\eta^{m,k}$ are continuous for $0 \leqslant \eta \leqslant 1$ and are continuously differentiable for $0 \leqslant \eta < 1$.

As in [4], the solution of problem (7), (8) can be obtained as the limit as $\varepsilon \to 0$ of the positive (for $0 \leqslant \eta < 1$) solutions of the system

$$\varepsilon w_{\eta\eta}^{m,k} + L_{m,k}(w) = 0, \qquad \varepsilon > 0, \quad m = 1, \ldots; \quad k = 0, 1, \ldots, [X/h]$$

with boundary conditions (8) whose existence can be proved with the aid of the Leray-Schauder theorem.

Lemma 2. For $0 \leqslant \eta \leqslant 1$ there exists a solution of the equation

$$\Lambda_n(Y) \equiv \nu Y^2 Y_{\eta\eta} - (n - 1/2)\,Y + (\eta - 1)\,nY_\eta = 0 \tag{9}$$

which satisfies the conditions

$$Y(1) = 0, \ \lambda_n(Y) \equiv (\nu Y Y_\eta + n)\,|_{\eta=0} = 0 \tag{10}$$

This solution has the following properties:

$$M_2(1-\eta)\,\sigma \leqslant Y(\eta) \leqslant M_1(1-\eta)\,\sigma \tag{11}$$

$$M_1(1-\eta)(\sigma - K) \leqslant Y(\eta) \quad \text{for } \eta_0 \leqslant \eta \leqslant 1 \tag{12}$$

$$-M_4\sigma \leqslant Y_\eta(\eta) \leqslant -M_3\sigma \tag{13}$$

$$|YY_{\eta\eta}| \leqslant M_5, \qquad YY_{\eta\eta} < -M_6 \tag{14}$$

$$(\sigma = \sqrt{-\ln \mu(1-\eta)}, \quad \mu = \text{const}, \quad 0 < \mu < 1)$$

Here μ is chosen in such a way that

$$\sigma^2|_{\eta=0} = 2n + {}^1/_2 + \delta, \quad \nu M_1{}^2 = 1, \quad \nu M_2{}^2 = {}^1/_2 - \delta$$
$$\delta,\ M_i,\ K,\ \eta_0 = \text{const} > 0; \quad i = 3, \ldots, 6$$

where δ is some small number.

Proof. Lemma 2 can be proved in the same way as Lemma 2 of [4]. We shall merely note the slight differences which arise for $n > 1$. It is easy to show that if

$$\varphi_1 = M_1(1-\eta)\sigma, \qquad \nu M_1^2 = 1, \qquad \sigma^2|_{\eta=0} = 2n + {}^1/_2 + \delta$$

then

$$\Lambda_n(\varphi_1) < 0 \quad \text{for} \quad 0 \leqslant \eta < 1, \qquad \lambda_n(\varphi_1) < 0$$

For this reason $Y(\eta) \leqslant \varphi_1(\eta)$. In similar fashion we can show that

$$Y(\eta) \geqslant \varphi_2(\eta) = M_2(1-\eta)\sigma, \qquad \nu M_2^2 = {}^1/_2 - \delta$$

for some sufficiently small constant M_2.

In order to obtain estimate (14) we must refine the lower estimate for $Y(\eta)$ in the neighborhood of $\eta = 1$. Let $\varphi_3(\eta) = M_1(1-\eta)(\sigma - K)$. Let us show that the constants $K > 0$ and $0 < \eta_0 < 1$ can be chosen in such a way that the inequality $Y(\eta) \geqslant \varphi_3$ is fulfilled for $\eta_0 \leqslant \eta \leqslant 1$. It is easy to see that

$$\Lambda_n(\varphi_3) = M_1(1-\eta)\left[\frac{K}{2}\left(1 - \frac{K}{\sigma}\right) - \frac{n}{2\sigma} - \frac{1}{4} + \frac{K}{2\sigma}\left(1 - \frac{K}{2\sigma}\right)\right]$$

Let the inequalities

$$K/\sigma < 1, \qquad 1 - K/\sigma \geqslant d, \qquad d = \text{const} > 0, \qquad M_1 d \leqslant M_2$$

be fulfilled for $\eta_0 \leqslant \eta \leqslant 1$.

Then, if K is large enough,

$$\Lambda_n(\varphi_3) > 0 \qquad \text{for} \quad \eta_0 \leqslant \eta < 1$$

Let us choose η_0 in such a way that

$$(1 - K/\sigma)|_{\eta=\eta_0} = d$$

Then

$$1 - K/\sigma \geqslant d, \qquad K/\sigma < 1 \qquad \text{for} \quad \eta_0 \leqslant \eta \leqslant 1$$

so that

$$\Lambda_n(\varphi_3) > 0 \qquad \text{for} \quad \eta_0 \leqslant \eta < 1$$

Further,

$$\varphi_3|_{\eta=\eta_0} = M_1 d(1-\eta_0)\sigma|_{\eta=\eta_0} \leqslant M_2(1-\eta_0)\sigma|_{\eta=\eta_0} \leqslant Y(\eta_0)$$

since $M_1 d \leqslant M_2$. Considering the equation

$$\Lambda_n(\varphi_3) - \Lambda_n(Y) = \Lambda_n(\varphi_3) \quad \text{for} \quad 0 \leqslant \eta < 1$$

for the difference $\varphi_3 - Y$ and bearing in mind the conditions

$$(\varphi_3 - Y)|_{\eta=1} = 0, \qquad (\varphi_3 - Y)|_{\eta=\eta_0} \leqslant 0$$

we find that

$$Y(\eta) \geqslant \varphi_3(\eta) \quad \text{for} \quad \eta_0 \leqslant \eta \leqslant 1$$

Now let us prove inequalities (13). We begin by introducing the symbol $z = Y_\eta$. Equation (9) yields the equation for z,

$$\nu Y^2 z_\eta + n(\eta - 1)z = (n - {}^1/_2)Y \tag{15}$$

Just as in Lemma 2 of [4], we can show with the aid of Eq. (15) that there exist constants M_3 and M_4 such that $-M_4\sigma \leqslant z \leqslant -M_3\sigma$ for $0 \leqslant \eta < 1$.

Estimates (11) and (13) imply that

$$|\nu Y Y_{\eta\eta}| \leqslant (n - {}^1/_2) + n(1-\eta)|Y_\eta| Y^{-1} \leqslant \nu M_5$$

Let us now show that

$$YY_{\eta\eta} < -M_6$$

Inequality (12) implies that there exists a sequence η^N which tends to unity as $N \to \infty$ and is such that

$$Y_\eta(\eta^N) \leqslant M_1(-\sigma + {}^1/_2\sigma^{-1})|_{\eta=\eta^N} + M_1 K$$

Hence, for a sufficiently small $1-\eta^N$ we have

$$\nu YY_{\eta\eta}\big|_{\eta=\eta^N} = [(n-{}^1/_2) + n(1-\eta)Y_\eta Y^{-1}]\big|_{\eta=\eta^N} \leqslant$$

$$\leqslant [(n-{}^1/_2) - n(1-{}^1/_2\sigma^{-2}) + nK/\sigma^{-1}]\big|_{\eta=\eta^N} < -M_7$$

$$(M_7 = \text{const} > 0)$$

Differentiating Eq. (9) with respect to η, we find that $R = YY_{\eta\eta}$ satisfies the equation

$$\Lambda^n(R) \equiv \nu YR_\eta + \nu Y_\eta R + n(\eta-1)RY^{-1} = -{}^1/_2 Y_\eta$$

Let $\Psi = -M_6$ and $0 < M_6 < M_7/\nu$. Then

$$\Lambda^n(R-\Psi) = -{}^1/_2 Y_\eta + M_6(\nu Y_\eta + (\eta-1)nY^{-1}) > 0 \tag{16}$$

for $0 \leqslant \eta < 1$ if M_6 is sufficiently small. From inequality (16) and the condition $(R-\Psi) < 0$ for $\eta = \eta^N$ we readily infer that $R - \Psi \leqslant 0$ for $0 \leqslant \eta \leqslant \eta^N$, which means that $R < -M_6$ for $0 \leqslant \eta < 1$.

Lemma 3. Let U_1, U_{1x}, $U_{1t}U_1^{-1}$, v_0 be bounded in D. The solutions of problem (7), (8) positive for $\eta < 1$ then satisfy the inequalities

$$mhY(1 - \alpha(mh)^{N-1}) \leqslant w^{m,k}(\eta) \leqslant mhY(1 + \beta(mh)^{N-1})$$

for $mh \leqslant \tau_0'$, $\tau_0' \leqslant \tau_0$.

Here α, β, τ'_0 are some positive constants independent of h.

Proof. Let us compute $L_{m,k}(F_1)$, where

$$F_1^{m,k}(\eta) = mhY(\eta)(1+\beta(mh)^{N-1})$$

For $m \geqslant 1$ we have

$$L_{m,k}(F_1) = (1+\beta(mh)^{N-1})[(mh)^3(\nu Y^2 Y_{\eta\eta} - (n-{}^1/_2)Y + (\eta-1)nY_\eta) + (mh)^3(2 +$$
$$+\beta(mh)^{N-1})\beta(mh)^{N-1}\nu Y^2 Y_{\eta\eta} + A_1^{m-1,k}((m-1)h)^{2N} mhY_\eta + B_1^{m-1,k}((m-1)h)^{2N} mhY] -$$
$$-h^{-1}(mh)^3(n-{}^1/_2)(m-1)hY\beta((mh)^{N-1} - ((m-1)h)^{N-1})$$

Since $YY_{\eta\eta} < -M_6$, it follows that $L_{m,k}(F_1) < 0$ for $\eta < 1$ provided that $\beta > 0$ is sufficiently large, that $mh \leqslant \tau_0'$, and that τ_0' is sufficiently small. The constants β and τ_0' do not depend on h.

Now let us compute $\lambda_{m,k}(F_1)$. We have

$$\lambda_{m,k}(F_1) = \{(mh)^2(\nu YY_\eta + n) + \nu(mh)^{N+1}(2+\beta(mh)^{N-1})\beta YY_\eta -$$
$$- v_0^{m-1,k}((m-1)h)^N mhY(1+\beta(mh)^{N-1}) + C_1^{m-1,k}((m-1)h)^{2N}\}|_{\eta=0} < 0$$

$$\left(N = 1 + \frac{1}{2n-1}\right)$$

provided that $\beta > 0$ is sufficiently large, that $mh \leqslant \tau_0'$, and that τ_0' is sufficiently small. The inequalities

$$L_{m,k}(F_1) - L_{m,k}(w) \leqslant 0, \quad \frac{1}{F_1^{m,k}}\lambda_{m,k}(F_1) - \frac{1}{w^{m,k}}\lambda_{m,k}(w) < 0$$

and the conditions

$$F_1^{0,k} - w^{0,k} = 0, \quad (F_1^{m,k} - w^{m,k})\big|_{\eta=1} = 0$$

imply that

$$w^{m,k} \leqslant mh(1+\beta(mh)^{N-1})Y \quad \text{for} \quad mh \leqslant \tau_0'$$

In similar fashion we can show that

$$w^{m,k} \geqslant mh(1-\alpha(mh)^{N-1})Y \quad \text{for} \quad mh \leqslant \tau_0'$$

if α is sufficiently large.

Lemma 4. Let U_1, U_{1x}, $U_{1t}U^{-1}$, v_0 have bounded derivatives with respect to ξ and τ. The following inequalities are then fulfilled for the solution $w^{m,k}$ of problem (7),(8) for $mh \leqslant \tau_1$:

$$Y_\eta(\eta)(mh)(1+\alpha_1(mh)^{N-1}) \leqslant w_\eta^{m,k}(\eta) \leqslant Y_\eta(\eta)(mh)(1-\beta_1(mh)^{N-1})$$

$$|h^{-1}(w^{m,k}-w^{m-1,k})| \leqslant (1+\varepsilon_1)Y, \; |h^{-1}(w^{m,k}-w^{m,k-1})| \leqslant mhY\left(N=1+\frac{1}{2n-1}\right)$$

$$|w^{m,k}w_{\eta\eta}^{m,k}| \leqslant K_1(mh)^2, \quad w^{m,k}w_{\eta\eta}^{m,k} < -K_2(mh)^2$$

Here $\tau_1, \alpha_1, \beta_1, K_1, K_2, \varepsilon_1 = \text{const} > 0$. These constants do not depend on h, and ε_1 can be chosen arbitrarily small.

This lemma can be proved exactly as Lemma 4 of [4].

Lemmas 3 and 4 directly imply the following theorem.

Theorem 1. Let the assumptions of Lemmas 3 and 4 concerning U_1 and v_0 be fulfilled. The solution w of problem (5),(6) then exists in the domain $\Omega_{\tau_1}\{0 \leqslant \tau \leqslant \tau_1,$ $0 \leqslant \xi \leqslant X, \quad 0 \leqslant \eta < 1\}$, where τ_1 depends on the functions U and v_0; this solution has the following properties:

the function $w(\tau, \xi, \eta)$ is continuous in the domain Ω_{τ_1} and

$$\tau Y(\eta)(1-\alpha\tau^{N-1}) \leqslant w(\tau, \xi, \eta) \leqslant \tau Y(\eta)(1+\beta\tau^{N-1}) \tag{17}$$

the derivative w_η is continuous in η for $0 \leqslant \eta < 1$ and

$$\tau Y_\eta(\eta)(1+\alpha_1\tau^{N-1}) \leqslant w_\eta(\tau, \xi, \eta) \leqslant \tau Y_\eta(\eta)(1-\beta_1\tau^{N-1})$$

the derivatives w_ξ, w_τ, $ww_{\eta\eta}$ are bounded in the domain Ω_{τ_1} and

$$|w_\xi| \leqslant \tau Y, \quad |w_\tau| \leqslant (1+\varepsilon_1)Y, \quad ww_{\eta\eta} < -K_2\tau^2$$

where the function w satisfies Eq. (5) almost everywhere in Ω_{τ_1}. The constants α, β, α_1, β_1, K_2, ε_1 are positive and depend on U, v_0 X.

Theorem 2. The solution of problem (5),(6) is unique in the class of functions $w \geqslant 0$ which: (a) are continuous in the domain Ω_{τ_1}; (b) have a derivative w_η continuous in η for $\eta = 0$; (c) satisfy Eq. (5) and conditions (6) almost everywhere; (d) are such that $w_\eta, w_\xi, w_\tau, w_{\eta\eta}$ are integrable within any interior subdomain of Ω_{τ_1}, $w_{\eta\eta} \leqslant 0$ where the functions $w_{1\eta}w$, $w_1 w_\eta$, w / w_1 are bounded and w_1 is the solution of problem (5),(6) constructed in Theorem 1.

This theorem can be proved by the procedure used to prove the uniqueness of the solution of problem (5),(6) for $n = 1$ in Theorem 1 of [4].

Let us now construct an expansion in powers of t asymptotic as $t \to 0$ for the solution w of problem (5),(6). The number of terms in the expansion (which we denote by q) is arbitrary. We shall also estimate the remainder term of the series. Let us begin with the case where

$$U(t, x) = t^n U_1(x), \quad v_0 \equiv 0, \quad n \geqslant 1$$

Under this assumption Eq. (5) and condition (6) become

$$\nu w^2 w_{\eta\eta} - \tau^3 (n - 1/2) w_\tau + n(\eta - 1)\tau^2 w_\eta - \eta U_1(\xi)\tau^{3N} w_\xi + \\ + (\eta^2 - 1) U_{1x}\tau^{3N} w_\eta - \eta U_{1x}(\xi)\tau^{3N} w = 0 \qquad (18)$$

$$\left(N = 1 + \frac{1}{2n-1}\right)$$

$$w|_{\tau=0} = 0, \quad w|_{\eta=1} = 0, \quad (\nu w w_\eta + n\tau^2 + U_{1x}(\xi)\tau^{3N})|_{\eta=0} = 0 \qquad (19)$$

Lemma 5. Let $U_1(x)$ have a bounded derivative of order $q+1$ for $0 \leqslant x \leqslant X$. The system of ordinary differential equations for $0 \leqslant \eta < 1$ for the functions $Y_i(\xi, \eta)$, $i = 1, \ldots, q$ which depend on the parameter ξ $(0 \leqslant \xi \leqslant X)$, i.e. the system of ordinary differential equations

$$L_i(Y) \equiv \nu Y_0^2 Y_{i\eta\eta} + n(\eta - 1) Y_{i\eta} + 2\nu Y_0 Y_{0\eta\eta} Y_i -$$

$$- (n - 1/2)\left(1 + i\frac{(2n+2)}{(2n-1)}\right) Y_i + \sum_{\substack{l+s+\rho=i \\ l\neq i,\ s\neq i,\ \rho\neq i}} \nu Y_l Y_s Y_{\rho\eta\eta} -$$

$$- \eta U_1(\xi) Y_{(i-1)\xi} + (\eta^2 - 1) U_{1x}(\xi) Y_{(i-1)\eta} - \eta U_{1x}(\xi) Y_{(i-1)} = 0 \qquad (20)$$

with the boundary conditions

$$Y_i|_{\eta=1} = 0, \quad \left(\nu Y_0 Y_{i\eta} + \nu Y_{0\eta} Y_i + \nu \sum_{\substack{l+s=i \\ s\neq i,\ l\neq i}} Y_l Y_{s\eta}\right)\Big|_{\eta=0} = 0 \qquad (21)$$

if $i = 2, \ldots, q$, and

$$Y_1|_{\eta=1} = 0, \quad (\nu Y_0 Y_{1\eta} + \nu Y_{0\eta} Y_1 + U_{1x}(\xi))|_{\eta=0} = 0 \qquad (22)$$

where $Y_0(\eta)$ is the solution $Y(\eta)$ of problem (9), (10), has a unique solution. This solution has the following properties:

$$|Y_i| \leqslant N_i (1-\eta)\sigma, \quad |Y_{i\eta}| \leqslant C_i\sigma, \quad |Y_0 Y_{i\eta\eta}| \leqslant R_i \qquad (23)$$

$$\left|\frac{\partial^s Y_i}{\partial \xi^s}\right| \leqslant N_{i,s}(1-\eta)\sigma, \quad \left|\frac{\partial^l Y_{i\eta}}{\partial \xi^l}\right| \leqslant C_{i,l}\sigma, \quad \left|Y_0 \frac{\partial^l Y_{i\eta\eta}}{\partial \xi^l}\right| \leqslant R_{i,l} \qquad (24)$$

for $s \leqslant q - i + 1$, $l \leqslant q - i$. The constants $N_i, C_i, R_i, N_{is}, C_{i,l}, R_{i,l}$ do not depend on ξ.

Proof. Let us first assume that the solutions $Y_i (i = 1, \ldots, q)$ of problem (20), (21), (22) exist and prove estimates (23), (24) for these solutions. These estimates are valid for $Y_0 = Y$ by virtue of Lemma 2. Let us verify them for Y_1 and then, assuming that they are fulfilled for $i \leqslant \rho - 1$, prove them for $i = \rho$, $\rho \leqslant q$. For Y_1 we have

$$L_1'(Y_1) \equiv \nu Y_0^2 Y_{1\eta\eta} + n(\eta - 1) Y_{1\eta} + 2\nu Y_0 Y_{0\eta\eta} Y_1 -$$

$$- (n - 1/2)\left(1 + \frac{2n+2}{2n-1}\right) Y_1 + (\eta^2 - 1) U_{1x} Y_{0\eta} - \eta U_{1x} Y_0 = 0 \qquad (25)$$

Moreover, boundary conditions (22) are fulfilled. Let us introduce the notation

$$L_i'(Y_i) \equiv \nu Y_0^2 Y_{i\eta\eta} + n(\eta - 1) Y_{i\eta} + 2\nu Y_0 Y_{0\eta\eta} Y_i - (n - 1/2)\left(1 + \frac{i(2n+2)}{2n-1}\right) Y_i$$

$$\lambda'(Y_i) \equiv (\nu Y_0 Y_{i\eta} + \nu Y_{0\eta} Y_i)|_{\eta=0}, \quad i = 1, \ldots, q$$

Let $\Psi_1 = N_1(1-\eta)\sigma$. Then

$$L_1'(\Psi_1) = N_1\left\{\nu Y_0^2\left[-\frac{1}{2\sigma(1-\eta)} - \frac{1}{4\sigma^3(1-\eta)}\right] + n(\eta-1)\left(-\sigma+\frac{1}{2\sigma}\right) + \right.$$
$$\left. + 2\nu Y_0 Y_{0\eta\eta}(1-\eta)\sigma - (n-{}^1/_2)\left(1+\frac{2n+2}{2n-1}\right)(1-\eta)\sigma\right\} \leqslant -N_1\gamma_1(1-\eta)\sigma$$
$$(N_1,\ \gamma_1 = \text{const} > 0)$$
$$\lambda'(\Psi_1) = N_1\left\{\nu Y_0\left(-\sigma+\frac{1}{2\sigma}\right) + \nu Y_{0\eta}\sigma\right\}\Big|_{\eta=0} \leqslant -N_1\gamma_2$$
$$(\gamma_2 = \text{const} > 0)$$

This implies that $L_1'(\Psi_1 \pm Y_1) < 0$ for $0 \leqslant \eta < 1$ and $\lambda'(\Psi_1 \pm Y_1) < 0$ if N_1 is sufficiently large. It is clear that $(\Psi \pm Y_1)|_{\eta=1} = 0$. Hence, by virtue of the maximum principle, $\Psi_1 \pm Y_1 \geqslant 0$ and $|Y_1| \leqslant \Psi_1$ for $0 \leqslant \eta \leqslant 1$.

Differentiating Eq. (25) s times with respect to ξ, we obtain an equation for

$$\partial^s Y_1 / \partial\xi^s, \qquad 1 \leqslant s \leqslant q$$

It is easy to see that

$$L_1'\,(N_{1,s}(1-\eta)\sigma \pm \partial^s Y_1/\partial\xi^s) < 0 \quad \text{for} \quad 0 \leqslant \eta < 1$$
$$\lambda'\,(N_{1,s}(1-\eta)\sigma \pm \partial^s Y_1/\partial\xi^s) < 0$$

if $N_{1,s}$ is sufficiently large. Moreover, $\partial^s Y_1/\partial\xi^s = 0$ for $\eta = 1$. Hence,

$$|\,\partial^s Y_1/\partial\xi^s\,| \leqslant N_{1,s}(1-\eta)\sigma$$

Now let us obtain an estimate for $Y_{1\eta}$. Let $Y_{1\eta} = z_1$. Equation (25) yields an equation for z_1,

$$\Lambda'(z_1) \equiv \nu Y_0^2 z_{1\eta} + n(\eta-1)z_1 = -2\nu Y_0 Y_{0\eta\eta} Y_1 +$$
$$+ (n-{}^1/_2)\left(1+\frac{2n+2}{2n-1}\right)Y_1 - (\eta^2-1)U_{1x}Y_{0\eta} + \eta U_{1x}Y_0 \tag{26}$$

Let $\Phi_1 = C_1\sigma$. Then

$$\Lambda'(\Phi_1) = C_1\,[\nu Y_0^2/2(1-\eta)\sigma + n(\eta-1)\sigma] \leqslant C_1(1-\eta)\sigma({}^1/_2 - n) \leqslant$$
$$\leqslant -C_1\gamma_3(1-\eta)\sigma,\ C_1,\ \gamma_3 = \text{const} > 0$$

Hence $\Lambda'(\Phi_1 \pm z_1) < 0$ for $0 \leqslant \eta < 1$ if C_1 is sufficiently large. Estimate (23) for Y_1 implies that there exist sequences η_N^+ and η_N^- such that $\eta_N^{\pm} \to 1$ as $N \to \infty$, and that for some $C_1 > 0$ we have

$$(z_1 - C_1\sigma)\Big|_{\eta=\eta_N^-} < 0, \qquad (z_1 + C_1\sigma)\Big|_{\eta=\eta_N^+} > 0$$

The inequalities

$$\Lambda'(\Phi_1 \pm z) \leqslant 0, \qquad (\Phi_1 \pm z_1)\Big|_{\eta=\eta_N^{\pm}} > 0$$

imply that $\Phi_1 \pm z_1$ cannot vanish for $0 \leqslant \eta < 1$. Hence, $\Phi_1 \pm z_1 > 0$ for $0 \leqslant \eta < 1$, which means that $|z_1| \leqslant C_1\sigma$.

Differentiating Eq. (26) l times ($l \leqslant q-1$) with respect to ξ, we obtain an equation for $\partial^l Y_{1\eta}/\partial\xi^l$ analogous to Eq. (26). The following estimate can be verified exactly as for $Y_{1\eta}$:

$$|\,\partial^l Y_{1\eta}/\partial\xi^l\,| \leqslant C_{1,l}\,\sigma \qquad \text{for} \quad 0 \leqslant \eta < 1$$

Dividing Eq. (25) by νY_0, we can use it to express $Y_0 Y_{\eta\eta}$. By virtue of the resulting estimates for Y_1, $Y_{1\eta}$ and properties of the function $Y_0 = Y$ established in Lemma 2, we find that

$$|\,Y_0 Y_{1\eta\eta}\,| \leqslant R_1, \qquad R_1 = \text{const} > 0$$

Differentiating Eq. (25) l times with respect to ξ, we obtain an equation which readily yields the estimate

$$|\,Y_0\partial^l Y_{1\eta\eta}/\partial\xi^l\,| \leqslant R_{1,l} \quad \text{for} \quad l \leqslant q-1$$

Estimates (23), (24) are therefore fulfilled for $i = 1$. Let us assume that they have been proved for $i \leqslant \rho - 1$ and prove them for $i = \rho$.

Let us consider $L_\rho{}'(\Psi_\rho)$, where $\Psi_\rho = N_\rho(1-\eta)\sigma$. It is clear that

$$L_\rho{}'(\Psi_\rho) \leqslant -N_\rho \varkappa_\rho (1-\eta)\sigma, \quad \lambda'(\Psi_\rho) \leqslant -N_\rho \varkappa_\rho{}', \quad \varkappa_\rho, \varkappa_\rho{}' = \mathrm{const} > 0$$

Hence, choosing our N_ρ sufficiently large, we find that

$$L_\rho{}'(\Psi_\rho \pm Y_\rho) < 0 \quad \text{for } 0 \leqslant \eta < 1, \qquad \lambda'(\Psi_\rho \pm Y_\rho) < 0$$

This implies that $|Y_\rho| \leqslant N_\rho(1-\eta)\sigma$, since $(\Psi_\rho \pm Y_\rho)|_{\eta=1} = 0$.

Differentiating Eq. (20) s times ($s \leqslant q - \rho + 1$) with respect to ξ, we obtain an equation for $\partial^s Y_\rho / \partial \xi^s$. Recalling the hypothesis of the induction, we obtain estimate (24) in the form

$$|\partial^s Y_\rho / \partial \xi^s| \leqslant N_{\rho, s}(1-\eta)\sigma$$

exactly as we obtained estimate (23) for Y_ρ. Furthermore, estimates (23), (24) for

$$Y_{\rho\eta}, \quad Y_0 Y_{\rho\eta\eta}, \quad \partial^l Y_{\rho\eta}/\partial\xi^l, \quad Y_0 \partial^l Y_{\rho\eta\eta}/\partial\xi^l \quad \text{for } l \leqslant q-\rho$$

can be justified exactly as for $\rho = 1$.

The existence of the solution of system (20) with conditions (21), (22) can be proved as follows. Problem (20)–(22) is linear for Y_i ($i = 1, \ldots, q$). The existence of a solution for the system of equations $\varepsilon Y_{i\eta\eta} + L_i(Y_i) = 0$, $\varepsilon > 0$, with boundary conditions (21), (22) follows from its uniqueness, since in this case it is possible to construct the Green function for the operator

$$\varepsilon Y_{i\eta\eta} + L_i{}'(Y_i)$$

under the boundary conditions $Y_i|_{\eta=1} = 0$, $\lambda'(Y_i) = 0$.

The uniqueness of the solution of this problem follows from the maximum principle. The estimate $|Y_i| \leqslant C_i(1-\eta)\sigma$, uniform in ε for the solutions of the system $\varepsilon Y_{i\eta\eta} + L_i(Y_i) = 0$ with conditions (21), (22) is obtainable exactly in the same way as the estimate for the solution Y_1 of problem (25), (22). The derivative with respect to η of such a solution for $0 \leqslant \eta \leqslant 1 - \delta$, $\delta = \mathrm{const} > 0$, can be estimated uniformly in ε by making use of the first-order equations for $Y_{i\eta}$ obtained from the equations $\varepsilon Y_{i\eta\eta} + L_i(Y_i) = 0$ and boundary conditions (21), (22).

The derivatives $Y_{i\eta\eta}$ and $Y_{i\eta\eta\eta}$ can be estimated uniformly in ε for $0 \leqslant \eta \leqslant 1 - \delta$ by expressing them on the basis of the equations $\varepsilon Y_{i\eta\eta} + L_i(Y_i) = 0$ and the equations obtained by differentiating with respect to η. It is clear that these solutions converge uniformly to the solution of problem (20)–(22) for some sequence $\varepsilon \to 0$.

Theorem 3. Let $U(t, x) = t^n U_1(x)$, $(n-1)$ be any nonnegative number, let $v_0 \equiv 0$, and let U_1 have a bounded derivative of order $q+1$ for $0 \leqslant x \leqslant X$. The following estimate is then valid for $0 \leqslant \tau \leqslant \tau_{q+1}$ for the solution w of problem (5), (6) whose existence was proved in Theorem 1:

$$\left| w(\tau, \xi, \eta) - \sum_{i=0}^{q} Y_i(\xi, \eta)\tau^{1+i\gamma} \right| \leqslant M'_q \tau^{1+(q+1)\gamma} Y_0 \quad \left(\gamma = \frac{2n+2}{2n-1}\right) \tag{27}$$

Here $Y_i(\xi, \eta)$ are the solutions of system (20) with conditions (21), (22) and τ_{q+1} is some number which depends on $U_1(x)$, n, q; $M'_q = \mathrm{const} > 0$.

Proof. We begin by stipulating that

$$Y_*^{m,k} \equiv \sum_{i=0}^{q} Y_i^k (mh)^{1+i\gamma}, \qquad Y_i^k \equiv Y_i(kh, \eta)$$

$$W_*^{m,k} \equiv Y_*^{m,k}(1 + \beta_q (mh)^\varkappa + \mu_q h) \quad (\varkappa = (q+1)\gamma)$$

Let us estimate the difference $w^{m,k} - W_*^{m,k}$. To this end we compute $L_{m,k}(W_*)$. Recalling Eqs. (9) and (20), we obtain

$$L_{m,k}(W_*) = (1 + \beta_q(mh)^\varkappa + \mu_q h)\{((1 + \beta_q(mh)^\varkappa + \mu_q h)^2 - 1)\nu(Y_*^{m,k})^2 Y_{*\eta\eta}^{m,k} +$$

$$+ \nu \sum_{l+s+\rho \geqslant q+1} Y_l^k Y_s^k Y_{\rho\,\eta\eta}^{\;k}(mh)^{3+(l+s+\rho)\gamma} - \frac{1}{h}\eta U_1(kh)((m-1)h)^{2+\gamma}(mh)^{1+q\gamma} \times$$

$$\times (Y_q^k - Y_q^{k-1}) - \eta U_1(kh)\,[((m-1)h)^{2+\gamma}\sum_{i=1}^{q-1}\frac{1}{h}(Y_i^k - Y_i^{k-1})(mh)^{1+i\gamma} -$$

$$-(mh)^{2+\gamma}\sum_{i=1}^{q-1} Y_{i\xi}^k (mh)^{1+i\gamma}] - (mh)^3(n - 1/2)\sum_{i=1}^{q} Y_i^k[\frac{1}{h}(mh)^{1+i\gamma} -$$

$$-((m-1)h)^{1+i\gamma}/h - (1 + i\gamma)(mh)^{i\gamma}] + (\eta^2 - 1)U_{1x}(kh)Y_{*\eta}^{m,k}[((m-1)h)^{2+\gamma} - (mh)^{2+\gamma}] +$$

$$+(\eta^2 - 1)U_{1x}(kh)(mh)^{3+(q+1)\gamma}Y_{q\eta}^{\;k} - \eta U_{1x}(kh)Y_*^{m,k}[((m-1)h)^{2+\gamma} - (mh)^{2+\gamma}] -$$

$$-\eta U_{1x}(kh)(mh)^{3+(q+1)\gamma}Y_q^k\} - \frac{1}{h}(mh)^3(n - 1/2)\beta_q((mh)^\varkappa - ((m-1)h)^\varkappa)Y_*^{m-1,k} \qquad (28)$$

It is easy to see that $L_{m,k}(W_*) < 0$ for $0 \leqslant \eta < 1$ if M and β_q are sufficiently large, if $mh \leqslant \tau_{q+1}$ and if τ_{q+1} is sufficiently small, since for $0 \leqslant mh \leqslant \tau_{q+1}$ we have the inequality $Y_*^{m,k}Y_{*\eta\eta}^{m,k} < -M_8$, $M_8 = \text{const} > 0$, and the expression

$$\nu(Y_*^{m,k})^2 Y_{*\eta\eta}^{m,k}(2 + \beta_q(mh)^\varkappa + \mu_q h)(\beta_q(mh)^\varkappa + \mu_q h)$$

for $mh \leqslant \tau_{q+1}$ and for large β_q and μ_q is larger in absolute value than all the nonnegative terms occurring in the right side of Eq. (28).

Let us compute $\lambda_{m,k}(W_*)$. Recalling boundary conditions (21), (22), (10), we obtain

$$\lambda_{m,k}(W_*) = [\nu(1 + \beta_q(mh)^\varkappa + \mu_q h)^2 Y_*^{m,k}Y_{*\eta}^{m,k} +$$

$$+ n(mh)^2 + U_{1x}(kh)((m-1)h)^{2+\gamma}]\,|_{\eta=0} =$$

$$= [\nu(2 + \beta_q(mh)^\varkappa + \mu_q h)(\beta_q(mh)^\varkappa + \mu_q h)Y_*^{m,k}Y_{*\eta}^{m,k} +$$

$$+ \sum_{s+l \geqslant q+1} \nu Y_s^k Y_{l\eta}^{\;k}(mh)^{2+(s+l)\gamma} + U_{1x}(kh)(((m-1)h)^{2+\gamma} - (mh)^{2+\gamma})]\,|_{\eta=0}$$

Since $Y_{*\eta}^{m,k}\big|_{\eta=0} < -M_9$, $M_9 = \text{const} > 0$ for sufficiently small mh, it follows that by choosing sufficiently large β_q and μ_q, we can ensure that $\lambda_{m,k}(W_*) < 0$ for $mh \leqslant \tau_{q+1}$.

Let us consider $S^{m,k} = W_*^{m,k} - w^{m,k}$. The above inequalities imply that

$$L_{m,k}(W_*) - L_{m,k}(w) < 0 \quad \text{for } 0 \leqslant \eta < 1$$

$$(W_*^{m,k})^{-1}\lambda_{m,k}(W_*) - (w^{m,k})^{-1}\lambda_{m,k}(w) < 0$$

Hence,

$$\nu(w^{m,k})^2 S_{\eta\eta}^{m,k} - h^{-1}(mh)^3(n - 1/2)(S^{m,k} - S^{m-1,k}) - n(\eta - 1)(mh)^2 S_\eta^{m,k} -$$

$$- h^{-1}U_1(kh)\eta((m-1)h)^{2+\gamma}(S^{m,k} - S^{m,k-1}) +$$

$$+(\eta^2 - 1)U_{1x}(kh)((m-1)h)^{2+\gamma}S_\eta^{m,k} - \eta U_{1x}(kh)((m-1)h)^{2+\gamma}S^{m,k} + \nu(w^{m,k} +$$

$$+ W_*^{m,k})W_{*\eta\eta}^{m,k}S^{m,k} < 0 \quad \text{for } 0 \leqslant \eta < 1$$

$$[\nu S_\eta^{m,k} - (n(mh)^2 + U_{1x}(kh)(mh)^{2+\gamma})(w^{m,k}W_*^{m,k})^{-1}S^{m,k}]\big|_{\eta=0} < 0$$

It is clear that for sufficiently small mh the coefficients of $S^{m,k}$ in these inequalities are negative. Hence, by virtue of the maximum principle and the conditions $S^{m,k}(1) = 0$, $S^{0,k} = 0$ we have the inequalities $S^{m,k} \geqslant 0$ for $mh \leqslant \tau_{q+1}$, so that

$$w^{m,k} \leqslant Y_*^{m,k}(1 + \beta_q(mh)^\varkappa + \mu_q h)$$

In exactly the same way we can prove that

$$w^{m,k} \geqslant Y_*^{m,k}(1 - \alpha_q (mh)^\varkappa - \gamma_q h)$$

for $mh \leqslant \tau_{q+1}$ (τ_{q+1} is sufficiently small) and certain α_q and γ_q independent of h.

Taking the limit as $h \to 0$ in the resulting inequalities for $w^{m,k}$, we obtain (27). The theorem has been proved.

Now let us consider the general case. Let

$$U(t, x) = t^n U_1(t, x)$$

$$U_1(t, x) = \sum_{s=0}^{\rho_1} a_s(x) t^s + a_{\rho_1+1}(t, x), \quad |a_{\rho_1+1}| \leqslant c_1 t^{\rho_1+1}$$

$$U_{1x}(t, x) = \sum_{s=0}^{\rho_2} a_s'(x) t^s + a'_{\rho_2+1}(t, x), \quad |a'_{\rho_2+1}| \leqslant c_2 t^{\rho_2+1}$$

$$U_{1t}/U_1 = \sum_{s=0}^{\rho_3} \theta_s(x) t^s + \theta_{\rho_3+1}(t, x), \quad |\theta_{\rho_3+1}| \leqslant c_3 t^{\rho_3+1}$$

$$v_0(t, x) = \sum_{s=0}^{\rho_4} b_s(x) t^s + b_{\rho_4+1}(t, x), \quad |b_{\rho_4+1}| \leqslant c_4 t^{\rho_4+1} \tag{29}$$

Here ρ_1, ρ_2, ρ_3, ρ_4 are certain nonnegative integers. In order to construct an asymptotic expansion for the solution w of problem (5), (6) in this case, we consider the following system of ordinary differential equations for Y_i (ξ, η), $i = 1, \ldots, q$, which depend on the parameter ξ:

$$\nu Y_0^2 Y_{i\eta\eta} + (\eta - 1) n Y_{i\eta} + 2\nu Y_0 Y_{0\eta\eta} Y_i -$$
$$- (n - 1/2)(1 + i/(2n-1)) Y_i + \sum_{\substack{l+s+\rho=i \\ l\neq i,\, s\neq i,\, \rho\neq i}} \nu Y_l Y_s Y_{\rho\eta\eta} -$$
$$- \eta \sum_{2s+l+2n+2=i} a_s(\xi) Y_{l\xi} + (\eta^2 - 1) \sum_{2s+l+2n+2=i} a_s'(\xi) Y_{l\eta} -$$
$$- \eta \sum_{2s+l+2n+2=i} a_s'(\xi) Y_l + (\eta - 1) \sum_{2s+l+2=i} \theta_s(\xi) Y_{l\eta} - \sum_{2s+l+2=i} \theta_s(\xi) Y_l = 0$$
$$Y_0 = Y, \quad 0 \leqslant \eta < 1, \quad q \geqslant 1 \tag{30}$$

and the boundary conditions

$$Y_i|_{\eta=1} = 0, \quad \Big(\nu Y_0 Y_{i\eta} + \nu Y_{0\eta} Y_i + \nu \sum_{\substack{l+s=i \\ s\neq i,\, l\neq i}} Y_l Y_{s\eta} -$$
$$- \sum_{2s+l+1=i} b_s(\xi) Y_l + \alpha^i \theta_{i/2-1}(\xi) + \beta^i a'_{i/2-n-1}(\xi)\Big)\Big|_{\eta=0} = 0 \tag{31}$$

Here $\alpha^i = \beta^i = 1$ if i is even, and $\alpha^i = \beta^i = 0$ if i is odd.

Lemma 6. System of differential equations (30) with boundary conditions (31) has the solution Y_i $(i = 1, \ldots, q)$, with the properties

$$|Y_i| \leqslant N_i(1 - \eta)\sigma, \qquad |Y_{i\eta}| \leqslant C_i \sigma, \qquad |Y_0 Y_{i\eta\eta}| \leqslant R_i$$
$$\left|\frac{\partial Y_i}{\partial \xi}\right| \leqslant N_i'(1 - \eta)\sigma \qquad (i = 1, \ldots, q)$$

$$\left|\frac{\partial^2 Y_i}{\partial \xi^2}\right| \leqslant N_i''(1-\eta)\sigma \quad \text{for} \quad i \leqslant q-2n-2$$

$$(N_i,\ C_i,\ R_i,\ N_i',\ N_i'' = \text{const} > 0)$$

provided that

$$\rho_1 \geqslant [q/2]-(n+1), \qquad \rho_2 \geqslant [q/2]-(n+1)$$

$$\rho_3 \geqslant [q/2]-1, \qquad \rho_4 \geqslant [(q-1)/2]$$

in Eqs. (29), and also that the functions a_s, a_s', θ_s, b_s have bounded derivatives of up to the order $[q/(2n+2)]+1$ with respect to x.

This lemma can be proved in the same way as Lemma 5. We can use Lemma 6 to prove the following theorem.

Theorem 4. Let $U_1\ (t, x) = t^n\ U_1\ (t, x)$, let $n \geqslant 1$ be an integer, and let the conditions of Theorem 1 and Lemma 6 for $q \geqslant 0$ be fulfilled for $U_1\ (t, x)$ and $v_0\ (t, x)$. The following relation is then valid for $0 \leqslant \tau \leqslant \tau_{q+1}$ for the solution w of problem (5), (6) whose existence was proved in Theorem 1:

$$\left|w(\tau, \xi, \eta) - \sum_{i=0}^{q} Y_i(\xi, \eta)\, \tau^{1+i/(2n-1)}\right| \leqslant K_q' Y_0 \tau^{1+\frac{(q+1)}{(2n-1)}} \tag{32}$$

Here $Y_i(\xi,\eta)$ are the solutions of system (30) with conditions (31), and $K_q', \tau'_{q+1} = \text{const} > 0$.

Theorem 4 can be proved in the same way as Theorem 3. Here we have

$$Y_*^{m,k} = \sum_{i=0}^{q} Y_i(kh, \eta)(mh)^{1+i/(2n-1)}$$

$$W_*^{m,k} = Y_*^{m,k}(1+\beta_q(mh)^{(q+1)/(2n-1)} + \mu_q h^{1/(2n-1)})$$

The term of the form

$$\nu(Y_*^{m,k})^2\, Y_{*\eta\eta}^{m,k}[(1+\beta_q(mh)^{(q+1)/(2n-1)} + \mu_q h^{1/(2n-1)})^2 - 1] =$$
$$= \nu(Y_*^{m,k})^2\, Y_{*\eta\eta}^{m,k}(2+\beta_q(mh)^{(q+1)/(2n-1)} +$$
$$+ \mu_q h^{1/(2n-1)})(\beta_q(mh)^{(q+1)/(2n-1)} + \mu_q h\,(mh)^{1/(2n-1)})^{-1}$$

in the expression for $L_{m,k}(W_*)$ is then negative for sufficiently large μ_q, β_q; it is also larger in absolute value than all of the nonnegative terms appearing in the expression for $L_{m,k}(W_*)$ if $\eta < 1$ and $mh \leqslant \tau'_{q+1}$. Hence $L_{m,k}(W_*) < 0$ for $0 \leqslant \eta < 1$ and $mh \leqslant \tau_{q+1}'$. In the same way we can verify that $\lambda_{m,k}(W_*) < 0$ for sufficiently large β_q and μ_q if $mh \leqslant \tau'_{q+1}$ and if τ'_{q+1} is sufficiently small.

On the basis of the above theorems concerning the solution of problem (5), (6) we obtain the following theorem on the solution of problem (1), (2).

Theorem 5. Let

$$U\ (t, x) = t^n U_1\ (t, x) \quad (n \geqslant 1)$$

$$U\ (t, 0) = 0, \quad U_1\ (t, x) > 0 \quad \text{for } x > 0$$

where U_{1x}, U_{1t}/U_1, v_0 have bounded first-order derivatives with respect to t and x.

A solution u, v of problem (1), (2) then exists in the domain

$$D_{T_1}\{0 \leqslant t \leqslant \tau_1^{2/(2n-1)} = T_1,\ 0 \leqslant x \leqslant X,\ 0 \leqslant y < \infty\}$$

This solution has the following properties:

$$u / U,\ u_y t^n / U \text{ are bounded and continuous in } D_{T_1}$$

$$u(t, x, y) > 0 \text{ for } tx > 0, \quad \frac{u_y t^n}{U} > 0 \text{ for } t > 0, \quad \frac{u_y t^n}{U} \to 0 \text{ as } y \to 0$$

The derivatives $u_y, u_x, u_{yy}, u_t, v_y$ are bounded and continuous in y,

$$|u_y| \leqslant E_1 t^{n-1/2}, \quad |u_{yy}| \leqslant E_2 t^{n-1}, \quad |u_t| \leqslant E_3 t^{n-1}, \quad |u_x| \leqslant E_4 t^n$$

The function v is continuous in y and bounded for bounded y, and

$$t^{-n+1/2} u_{yx}, \quad t^{-n+3/2} u_{yt} \text{ are bounded for bounded } y$$

$$|u_{yyy}| \leqslant E_5 t^{n-3/2}, \quad E_i = \text{const} > 0$$

The equations of system (1) are satisfied almost everywhere in D_{T_1}. For this solution we have the estimates

$$\Phi^{-1}(yt^{-1/2}(1 - \alpha t^{1/2}))U \leqslant u \leqslant \Phi^{-1}(yt^{-1/2}(1 + \beta t^{1/2}))\, U \tag{33}$$

$$\alpha, \beta = \text{const} > 0$$

$$\Phi(\zeta) \equiv \int_0^\zeta (Y_0(s))^{-1} ds \qquad (\Phi^{-1} \text{ is the inverse of the function } \Phi)$$

$$U(1 - e^{-\nu_1}) \leqslant u \leqslant U(1 - e^{-\nu_2}) \tag{34}$$

$$\nu_2 = [M_1 y / (2t^{1/2}(1 - \beta t^{1/2}))]^2 + M_1 y \sqrt{-\ln \mu} / (t^{1/2}(1 - \beta t^{1/2}))$$

$$\nu_1 = [M_2 y / (2t^{1/2}(1 + \alpha t^{1/2}))]^2 + M_2 y \sqrt{-\ln \mu} / (t^{1/2}(1 + \alpha t^{1/2}))$$

$$\nu M_1^2 = 1, \quad \nu M_2^2 = 1/2 - \delta, \quad \delta = \text{const} > 0, \quad \varepsilon = \text{const} > 0$$

$$1 - \frac{u}{U} = \exp\left(-\frac{1}{4\nu t}[y^2 + O(y^{1+\varepsilon} t^{(1-\varepsilon/2)})]\right) \quad \text{for } y \to \infty,\ t \to 0$$

$$|u_y t^n / U - t^{n-1/2} Y_0(u / U)| \leqslant E_6 t^n Y_0(u / U).$$

$$|t^n u_{yy} / u_y - t^{n-1/2} Y_{0\eta}(u / U)| \leqslant E_7 t^n Y_{0\eta}(u / U)$$

$$t(u_{yyy} u_y - (u_{yy})^2)\, u_y^{-2} < -E_8, \quad E_i = \text{const} > 0$$

The solution u, v of problem (1), (2) is unique in the class of functions u, v for which $w = u_y t^n / U$ satisfies the conditions of Theorem 2.

Theorem 6. Let $U(t, x) = t^n U_1(x)$, let $v_0 \equiv 0$, and let $U_1(x)$ have a bounded derivative of the order $q + 1$. The following estimates are then valid for $0 \leqslant t \leqslant t_q$ for the solution u, v of problem (1), (2) obtained in Theorem 5:

$$\left| u_y t^n / U - t^{n-1/2} \sum_{i=0}^{q} Y_i(x, u / U)\, t^{i(n+1)} \right| \leqslant$$

$$\leqslant M_q' Y_0(u / U)\, t^{n-1/2+(q+1)(n+1)}, \qquad M_q' = \text{const} \tag{35}$$

where $Y_i(\xi, \eta)$, $i = 1, \ldots, q$ are the solutions of problem (20), (21), (22) and $Y_0(\eta) = Y(\eta)$ is the solution of problem (9), (10).

Specifically, estimate (35) yields a formula for the expansion of the quantity $u_y(t, x, 0)$ asymptotic as $t \to 0$, and an estimate of the remainder term.

$$\left| u_y(t, x, 0) - U_1(x) \sum_{i=1}^{q} Y_i(x, 0)\, t^{n-1/2+i(n+1)} \right| \leqslant$$

$$\leqslant M_q'' U_1(x)\, t^{n-1/2+(q+1)(n+1)}, \qquad M_q'' = \mathrm{const} \tag{36}$$

Theorem 7. Let the premises of Theorem 4 be fulfilled for $U(t, x)$ and $v_0(t, x)$. The following inequality is then fulfilled for $0 \leqslant t \leqslant t_q'$ for the solution u, v of problem (1), (2) obtained in Theorem 5:

$$\left| \frac{u_y t^n}{U} - \sum_{i=0}^{q} Y_i(x, u/U)\, t^{n-1/2+i/2} \right| \leqslant K_q' Y_0(u/U)\, t^{n+q/2} \tag{37}$$

where $Y_i(\xi, \eta)$ are the solutions of ordinary differential equations (30) with conditions (31); $Y_0(\eta) = Y(\eta)$, $K_q' = \mathrm{const} > 0$.

The following formula for the asymptotic expansion of $u_y(t, x, 0)$ (as $t \to 0$) and estimate of the remainder term are valid:

$$\left| u_y(t, x, 0) - U_1(t, x)\, t^{n-1/2} \sum_{i=0}^{q} Y_i(x, 0)\, t^{i/2} \right| \leqslant$$

$$\leqslant K_q'' U_1(t, x)\, t^{n+q/2}, \quad K_q'' = \mathrm{const} > 0 \tag{38}$$

Theorems 6 and 7 follow directly from Theorems 3 and 4.

The proof of Theorem 5 is similar to that of theorem 2 in [4]. The condition $w(\tau, \xi, \eta) = u_y t^n / U$ yields the following expression for determining $u(t, x, y)$:

$$y = t^n \int_0^{u^*} (w(t^{n-1/2}, x, s))^{-1}\, ds, \qquad u^* = u(t, x, y) / U(t, x) \tag{39}$$

Inequalities (17) yield the relations

$$\Phi(u/U)(1 + \beta t^{1/2})^{-1} \leqslant y t^{-1/2} \leqslant \Phi(u/U)(1 - \alpha t^{1/2})^{-1}, \quad \Phi(\zeta) = \int_0^{\zeta} \frac{ds}{Y_0(s)}$$

Let us denote the inverse of the function $\Phi(\xi)$ by $\Phi^{-1}(s)$. Then

$$U(t, x)\, \Phi^{-1}(y(1 - \alpha t^{1/2})\, t^{-1/2}) \leqslant u \leqslant U(t, x)\, \Phi^{-1}(y(1 + \beta t^{1/2}) t^{-1/2})$$

By virtue of Lemma 2 we have

$$\frac{2}{M_1}\left(\sqrt{-\ln(1-\zeta)} - \sqrt{-\ln \mu}\right) \leqslant \Phi(\zeta) \leqslant$$

$$\leqslant \frac{2}{M_2}\left(\sqrt{-\ln(1-\zeta)} - \sqrt{-\ln \mu}\right)$$

Estimates (34) are therefore valid for $u(t, x, y)$. Similarly, estimates (17), (11), (12) imply the relation

$$1 - \frac{u}{U} = \exp\left(-\frac{1}{4\nu t}[y^2 + O(y^{1+\varepsilon} t^{(1-\varepsilon)/2})]\right) \begin{pmatrix} y \to \infty \\ t \to 0 \end{pmatrix}$$

Here $\varepsilon > 0$ is an arbitrarily small number.

In the case where the premises of Theorem 3 are fulfilled we can set

$$\Phi_q(u/U, t, x) = \int_0^{u/U} \left(\sum_{i=0}^{q} Y_i(x, s)\, t^{n-1/2+i(n+1)} \right)^{-1} ds$$

to infer from Theorem 3 and relation (29) that

$$| y^{-1} \Phi_q(u/U, t, x)\, t^n - 1 | \leqslant E_9\, t^{(q+1)(n+1)} \tag{40}$$

Here $Y_i(\xi, \eta)$ are the solutions of system (20) with conditions (21), (22); $Y_0(\eta)$ is the solution of problem (9), (10); $E_9 = \text{const} > 0$.

If the premises of Theorem 4 are fulfilled, estimate (32) for w yields the following relation for $u(t, x, y)$. We write

$$\Phi_q^*(u/U, t, x) = \int_0^{u/U} \left(\sum_{i=0}^{q} Y_i(x, s)\, t^{n-1/2+i/2} \right)^{-1} ds$$

where $Y_i(\xi, \eta)$ are the solutions of system (30) with conditions (31). Then

$$|y^{-1}\Phi_q^*(u/U, t, x)\, t^n - 1| \leqslant E_{10}\, t^{1/2(q+1)}, \quad E_{10} = \text{const} > 0 \qquad (41)$$

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ASYMPTOTIC METHOD IN THE PROBLEM OF OSCILLATIONS OF A STRONGLY VISCOUS FLUID

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In [1] the authors have proved a theorem on the existence of solution of the Cauchy's problem for linearized equations corresponding to the problem of motion about a fixed point of a rigid body, with a cavity partially filled with a viscous incompressible fluid. In the case of small Reynolds numbers (high viscosity fluids), these equations will contain a small parameter $\varepsilon = \nu^{-1}$ and the Krylov-Bogoliubov asymptotic method given in [2] can be used to solve the system of Navier-Stokes equations. In the present paper we derive formulas for the corresponding approximate solutions. The case of a highly viscous fluid filling the cavity completely was investigated by Chernous'ko in [3 and 4].

1. Statement of the problem. We assume that a body with a cavity partially filled with a viscous incompressible fluid performs a given motion about a fixed point with an instantaneous angular velocity $\boldsymbol{\omega}$. It is required to determine the motion of fluid in the vessel. In the linearized formulation this problem reduces to solution of the following system of Navier-Stokes equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} = -\nabla q + \nu \Delta \mathbf{u}, \qquad \operatorname{div} \mathbf{u} = 0 \qquad (1.1)$$

in the region Ω filled with fluid in the state of equilibrium, with the boundary conditions